Tunneling and mode conversion of waves in the ion cyclotron frequency range in the planetary magnetospheres

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Introduction In addition to protons, heavy ion species such as Na^+, $\mathrm{He}^{(2)+}$, O^+, S^{2+}, ... are present in planetary environments. Narrow-band linearly polarized ultra-low frequency (ULF) waves $f \simeq 1$ Hz, having a resonant structure and a peak frequency between the cyclotron frequency of protons and heavy ions, have been detected in the magnetospheres of Earth and of Mercury [1–3]. Such wave events have been suggested to be driven by linear mode conversion (MC) of the fast magnetosonic waves (FW) at the ion-ion hybrid (IIH) resonances [4]. Since the resonant IIH frequency is linked to the plasma composition, solving the inverse problem allows one to infer the concentration of the heavy ions from the measured frequency spectra. In this paper, we identify numerically the conditions, when the MC efficiency is maximized in the magnetospheric plasmas of Earth and Mercury, and confirm analytical results derived in [5].

Conditions for efficient mode conversion In plasmas composed of two or more ion species (with different Z/A ratios), there is an additional frequency at which the FW is resonant, $\epsilon_1 = n_\parallel^2$. Here, $\epsilon_1(\omega) = 1 - \sum_s \omega_{ps}^2/(\omega^2 - \omega_{cs}^2)$ and $n_\parallel = ck_\parallel/\omega$ is the parallel (along the magnetic field) refractive index. This frequency ω_S is known as the IIH resonance and satisfies $\omega_{\mathrm{c}2} < \omega_S < \omega_{\mathrm{cH}}$ (subindex '2' refers to heavy ions). At the IIH resonant layer, the incoming FW is partially converted to a short wavelength mode. However, due to the intrinsic inhomogeneity of the magnetic field the IIH resonance is accompanied by the L-cutoff layer to the low magnetic field side, from which part of the FW power is reflected. The IIH resonance and L-cutoff form together the MC layer that is a barrier for the FW propagation. Only a fraction $\mathcal{T} = e^{-\pi\eta}$ of the incoming power tunnels through the layer, where η is the tunneling factor, roughly being a product of the perpendicular FW wave number and the MC layer width. In paper [5], we proved that waves with small $k_\parallel$ undergo almost total reflection and thus cannot be detected by the satellite. On the contrary, the FW critical parallel wave number $k_\parallel^*$ was derived and it was concluded that MC can be efficient only for waves with $k_\parallel/k_\parallel^* \lesssim 1$.

Typically, the concentration of heavy ions in magnetospheric plasmas is not known *a priori*. Here, we describe briefly how one can estimate it from the recorded resonant ULF spectrogram. For the measured wave frequency $f = f_{\mathrm{obs}}$, a set of $(k_\parallel, X_2)$–values satisfies the FW resonant condition $\epsilon_1 = n_\parallel^2$ at the given observation point

$$X_2 = X_{\mathrm{min}} + (X_{\mathrm{max}} - X_{\mathrm{min}})(1 - (k_\parallel/k_\parallel^*)^2),$$
$$X_{\mathrm{min}} = \frac{(f/f_{\mathrm{cH}})^2 - \mathcal{Z}_2^2}{\mathcal{Z}_1^2 - \mathcal{Z}_2^2}, X_{\mathrm{max}} = \frac{\mu X_{\mathrm{min}}}{1 + (\mu - 1)X_{\mathrm{min}}}, \tag{1}$$

where $X_2 = Z_2 n_2/n_e$ is the *unknown* concentration of heavy ions multiplied by their charge number, $\mathcal{Z}_1 = Z_1/A_1 = 1$ and $\mathcal{Z}_2 = Z_2/A_2$ is the ratio of the charge number to the atomic mass for protons and heavy ions, respectively, and $\mu = \mathcal{Z}_1/\mathcal{Z}_2$. The FW critical wave number depends on the plasma density $k_\parallel^* = \frac{\omega_{\rm pH}}{c}\frac{(f/f_{\rm cH})}{\sqrt{\mathcal{Z}_1+\mathcal{Z}_2}}$, with $\omega_{\rm pH} = \sqrt{4\pi n_e e^2/m_{\rm H}}$ being a reference proton plasma frequency. Note that $k_\parallel^*$ scales linearly with the wave frequency and is independent of the heavy ion concentration X_2 and the radius of the planet. As an *input* value for inferring the heavy ion concentration, we use the ratio of the resonant ULF frequency to the local gyrofrequency of protons $f/f_{\rm cH}$.

Figure 1(a) shows the $(k_\parallel, X_2)$–diagram computed for the conditions of Earth's ($R = 6.6R_{\rm E}$ and $n_e = 10\ {\rm cm}^{-3}$) and Mercury's ($R = 1.5R_{\rm M}$ and $n_e = 3\ {\rm cm}^{-3}$) magnetospheres, choosing the input frequency ratios $f/f_{\rm cH} \approx 0.44$ and 0.38, respectively [1, 2]. As follows from the figure, Eq. (1) reproduces excellently the numerical results. In addition, for every pair of $(k_\parallel, X_2)$ values we evaluated numerically the tunneling factor and the maximum MC efficiency $\mathcal{C}_{\rm max} = 4\mathcal{T}(1-\mathcal{T})$. The latter is represented by the colour of the points. The factor of 4 in the expression for $\mathcal{C}_{\rm max}$ appears due to the constructive/destructive interference when the right-hand polarized R-cutoff is also present in the plasma, making the enhancement of MC efficiency possible up to 100% [6]. Note that for low $k_\parallel$, the actual MC coefficient is given by the Budden expression, which is four times smaller, but then MC is negligible for both models.

For example, the Mariner 10 spacecraft detected a resonant linearly polarized ULF wave event at Mercury's magnetosphere (including a significant fraction of sodium ions Na^+) peaked at $f \approx 0.5$ Hz [2]. The local magnetic field strength was $B \approx 86$ nT and hence proton and sodium gyrofrequencies were $f_{\rm cH} \approx 1.31$ Hz and $f_{\rm c,Na^+} \approx 0.057$ Hz. One can consider two different extreme cases for the measured $f/f_{\rm cH} \approx 0.38$. If the detection of waves with $k_\parallel \simeq 0$ is assumed, then the IIH resonant frequency ω_S reduces to the well-known Buchsbaum two-ion hybrid frequency $\omega_{S0} = \omega_{\rm c2}\sqrt{\frac{1-(1-\mu)X_2}{1-(1-1/\mu)X_2}}$, which satisfies $\epsilon_1(\omega_{S0}) = 0$. However, for typical plasma conditions of Earth and Mercury, the MC layer is non-transparent $(\eta \gg 1)$ for such $k_\parallel$, and therefore the FW cannot tunnel through the barrier and undergo mode conversion [5]. Another limiting case corresponds to $k_\parallel = k_\parallel^*$ and $X_2 = X_{\rm min}$. For these parameters, the IIH resonance, L-cutoff and R-cutoff intersect at the observation point such that the MC layer width and accordingly the tunneling factor go to zero. This occurs for $\epsilon_2(\omega_{\rm cross}) = 0$, where $\epsilon_2(\omega) = -\sum_s(\omega_{cs}/\omega)\,\omega_{ps}^2/(\omega^2-\omega_{cs}^2)$ is a non-diagonal dielectric tensor element, and is typically referred to as the crossover condition [7]. The crossover frequency in a two-ion species plasmas is linked to the heavy ion concentration as $\omega_{\rm cross} = \omega_{\rm c2}\sqrt{1+(\mu^2-1)X_2}$.

Thus, for the measured frequency ratio for Mercury's plasma ($f/f_{\rm cH} \approx 0.38$), one gets $X_{\rm min}[\mathrm{Na}^+] = 14.3\%$ and $X_{\rm max}[\mathrm{Na}^+] = 79.3\%$. For the frequency ratio $f/f_{\rm cH} \approx 0.44$ reported for Earth in [1] ($f_{\rm obs} \approx 1.0$ Hz, $f_{\rm cH} \approx 2.28$ Hz), Eq. (1) implies that $X_{\rm min}[\mathrm{He}^+] = 14.0\%$ and

$X_{\max}[\mathrm{He}^+] = 39.4\%$. These values, corresponding to the Buchsbaum ($k_\| = 0$, $\eta \gg 1$) and the crossover ($k_\| = k_\|^*$, $\eta = 0$) conditions, yield the *upper* and the *lower* estimates for the heavy ion concentration, respectively.

Since efficient MC requires $0.05 < \eta < 0.61$ and is maximized at $\eta \approx 0.22$ ($\mathcal{T} = 1/2$) [5], one can conclude that the FW parallel wave number should be close to, but somewhat below, $k_\|^*$ (see also Fig. 1(b)). For $X_2 = X_{\min}$, decreasing $k_\|$ will result in a non-zero MC layer width. Simultaneously, the resonant layer will be shifted to the higher magnetic field side relative to the observation point. As a result, the heavy ion concentration should be somewhat larger than $X_{\min}$ to displace the resonant layer in the opposite direction and locate it back at the observation point (for the fixed wave frequency). This opposite shift of the IIH resonant layer due to the change of $k_\|$ and X_2 is captured by Eq. (1): for any heavy ion concentration within the range $X_{\min} \leq X_2 \leq X_{\max}$, there exists a particular wave number $k_\|$, for which the IIH resonance is located at the point of observation.

As follows from Fig. 1(a), the maximum MC efficiency ($\mathcal{C}_{\max} = 1$) is reached at $k_\|/k_\|^* \approx 0.93$ and 0.98 for the plasmas of Mercury and Earth, respectively. Once this wave number ratio is known, the heavy ion concentration can be easily evaluated. For Earth, Eq. (1) yields $X[\mathrm{He}^+] \approx 15.1\%$ ($f_{\mathrm{cross}} \approx 1.03$ Hz), while for Mercury $X[\mathrm{Na}^+] \approx 23.3\%$ ($f_{\mathrm{cross}} \approx 0.63$ Hz). The ratio of the IIH resonant to the crossover frequency is also linked to the $k_\|/k_\|^*$ value

$$p = \omega_S/\omega_{\mathrm{cross}} = \left[1 + G(x)\left(1 - (k_\|/k_\|^*)^2\right)\right]^{-1/2}, \quad G(x) = \frac{(x^2 - \mathcal{Z}_2^2)(\mathcal{Z}_1^2 - x^2)}{x^2(\mathcal{Z}_1\mathcal{Z}_2 + x^2)}, \tag{2}$$

where $x = f/f_{\mathrm{cH}}$. Equation (2) predicts $p \approx 0.97$ for the magnetosphere of Earth, whereas for Mercury $p \approx 0.79$, in agreement with the values reported in [8]. Vice versa, if the $\omega_S/\omega_{\mathrm{cross}}$ ratio is known, by combining Eqs. (1) and (2), the next-order correction to the heavy ion concentration can be explicitly written as $X_2/X_{\min} = (x^2/p^2 - \mathcal{Z}_2^2)/(x^2 - \mathcal{Z}_2^2) \gtrsim 1$. For Mercury, this can be simplified further since $\mathcal{Z}_2 = 1/23 \ll 1$, and $X[\mathrm{Na}^+] \approx X_{\min}[\mathrm{Na}^+]/p^2$.

As a function of f/f_{cH}, $G(x)$ in Eq. (2) maximizes at $\hat{x} = \mathcal{Z}_1/\sqrt{\mu^{3/2} - \mu + \mu^{1/2}}$ and equals to $\hat{G} = (1 + 1/\mu)(\sqrt{\mu} - 1)^2$. This yields $\hat{G}_{\mathrm{Earth}} = 1.25$ and substantially larger quantity for Mercury $\hat{G}_{\mathrm{Mercury}} \approx 15.0$. For the measured f/f_{cH} values, this G–term is equally – as the difference in the computed $k_\|/k_\|^*$ values – contributing to the fact why the $\omega_S/\omega_{\mathrm{cross}}$ ratio is closer to one for Earth than for Mercury ($G_{\mathrm{Earth}}(0.44) \approx 1.2$ vs. $G_{\mathrm{Mercury}}(0.38) \approx 4.5$). Note that whereas for Mercury's plasma the resonant frequency ω_S is close, but somewhat below, the crossover frequency ($p \approx 0.8$) (see Fig. 1(c)), for the magnetosphere of Earth the resonant IIH frequency is particularly close to the crossover frequency (Fig. 1(d)). This potentially allows an accurate determination of $X[\mathrm{He}^+]$ by matching f_{obs} to f_{cross}, without a need to go to the second-order approximation as in case of Mercury. However, the crossover frequency is also upshifted/downshifted accounting for additional ion species (e.g. O^+). This effect seems to be more important for the Earth's magnetosphere than the correction due to the finite p–value and was considered in [5].

Conclusions We have complemented previous analytical results derived in [5] by evaluating numerically the tunneling factor and the maximum MC efficiency $\mathcal{C}_{\max}$ for the magnetospheres of Earth and of Mercury. Efficient MC is shown to occur in a narrow range of $k_\parallel$ close to the FW critical wave number $k_\parallel^*$, which depends mainly on the plasma density and the $f/f_{\rm cH}$ ratio, and is independent of the radius of the planet and the heavy ion concentration. For the detected wave event in Mercury's magnetosphere, ω_S is confirmed to be close, but somewhat below, the crossover frequency $(p \approx 0.8)$. For the conditions of the magnetosphere of Earth, the resonant IIH frequency is calculated to be particularly close to the crossover frequency for a wide range of parameters $(p \approx 0.97)$, potentially making an accurate estimate of $X[\mathrm{He}^+]$ from the measured resonant linearly polarized ULF signal possible.

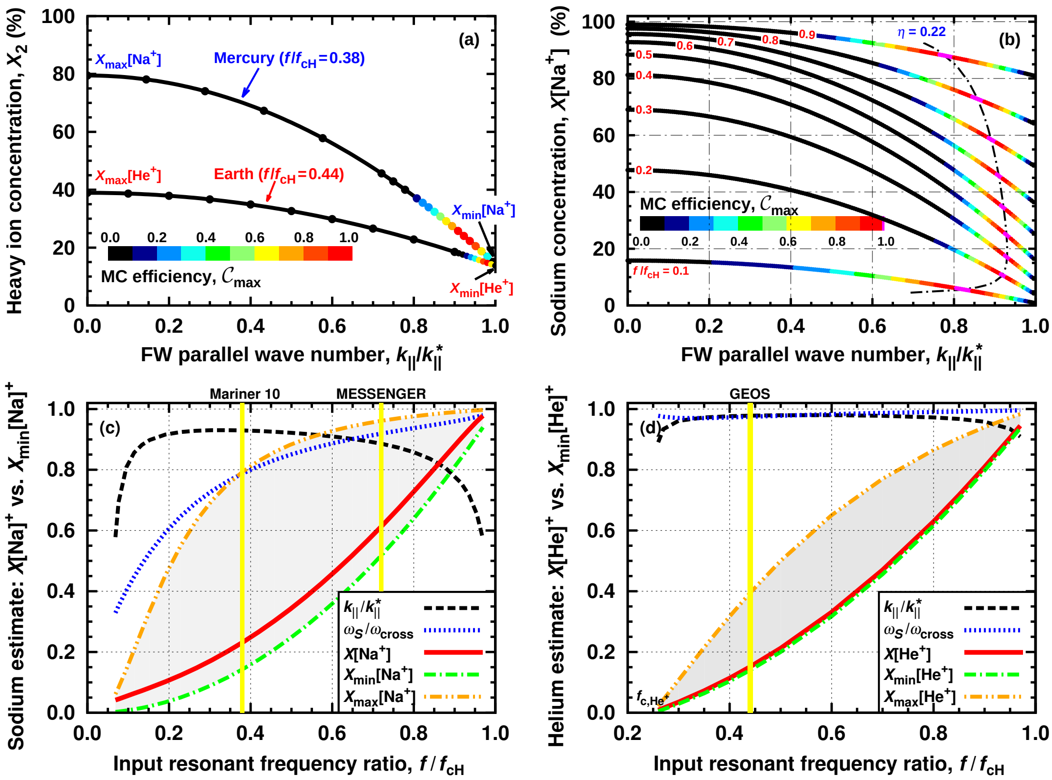

Figure 1: (a) Heavy ion concentration vs. the FW parallel wave number for the fixed IIH resonance location. The colour of the points represents the maximum MC efficiency, $\mathcal{C}_{\max} = 4\mathcal{T}(1-\mathcal{T})$. (b) The $(k_\parallel, X[\mathrm{Na}^+])$–diagram for the different $f/f_{\rm cH}$ ratios for Mercury's plasma. (c) and (d) Sodium and helium concentrations vs. $f/f_{\rm cH}$ for the magnetospheres of Mercury and Earth, respectively.

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