# Analytical models for skewness-kurtosis relations in turbulent transport

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Observations of low frequency turbulent fluctuations in magnetized plasmas demonstrated that under a wide range of parameters, these fluctuations contain a significant contribution from coherent structures. Observations in the magnetized toroidal Blaamann plasma device at the University of Tromsø, in particular, demonstrated that significant parts of the anomalous plasma losses were due to irregular plasma bursts associated with large spatial scales comparable to the minor radius of the device.

A diagram of the experimental set-up is shown in Fig. 1, where a movable probe-head is to the right. The plasma is characterized by a radial electric field $\mathbf{E}_0$ which together with the confining magnetic field $\mathbf{B}$ gives rise to a $\mathbf{E}_0 \times \mathbf{B}/B^2$-rotation of the plasma column as indicated by an arrow [1]. Using the movable probe we detect the fluctuating plasma density $\widetilde{n}$ and the $x$-component of the fluctuating $\widetilde{\mathbf{E}} \times \mathbf{B}/B^2$-velocity at selected positions along the $x$-axis as indicated in Fig. 1. The data are stored and used to construct the plasma flux signal $\Gamma(t) \equiv \widetilde{n}\widetilde{E}_y/B$. The average $\langle\Gamma\rangle$ gives the $x$-component of the net plasma flux. The signal contains a fluctuating component where the probability density can be obtained empirically. The average $\langle\Gamma\rangle$ corresponds to a net flux out of the plasma column for $x < 0$ as well as $x > 0$, see Fig. 1. In addition to the average, we characterize the probability density by its lowest order averages, i.e. the mean-square fluctuation level $\sigma^2$, the skewness $S$, and the kurtosis $K$. As for many other cases [2, 3, 4] we find that also here there is a near parabolic empirical relation between skewness $S$ and kurtosis $K$, as shown in Fig. 2 for two examples of discharges (argon and helium).

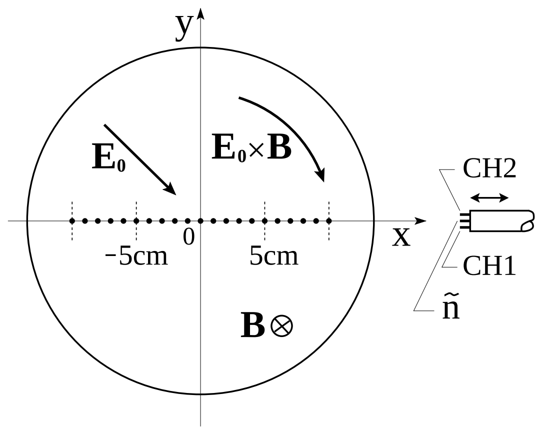


Figure 1: *Schematic illustration of the positions for data acquisition by the moving probe in a cross section of the Blaamann torus. An electric field component is obtained from the potential difference between probes CH1 and CH2 on the movable probe. Positions for data acquisition are shown with small filled circles along the x-axis.*

A simple, yet realistic, model is presented in terms of a superposition of non-overlapping coherent structures placed with random intervals. The model predicts the probability density of the turbulent plasma flux; in particular it includes a relation between the skewness, $S$, and the kurtosis, $K$, of the randomly varying flux. Although not exact, this skewness-kurtosis relation has the form of a parabolic variation $K = aS^2 + b$, where $a$ and $b$ are constants to be fitted for any particular problem.

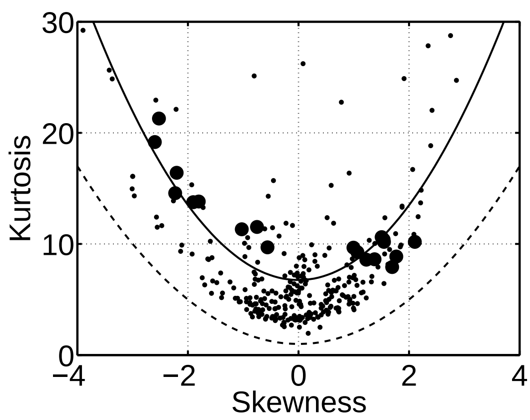


Figure 2: *Scatter plot for skewness and kurtosis obtained at the spatial sampling positions of the flux signal. Large filled circles refer to the data for helium used in the present study, while the small dots are for an argon plasma. For reference we show the relation* $K = S^2 + 1$ *with a dashed line. The full line shows the best fit for the helium data,* $K = 1.7S^2 + 6.8$. *Negative* $S$ *are found for* $x < 0$, *positive* $S$ *for* $x > 0$, *see also Fig. 1.*

In its simplest version, the model contains two signal levels only. We assume that the flux is "burst-like", i.e. it is either vanishing or it assumes a constant positive value $\gamma > 0$ in a short time interval $\Delta\tau$. The time variation of the flux event thus has a so called "top-hat" shape. The random process is assumed to be time stationary, and the probability for encountering a plasma burst at some position is the same for all times. At some fixed spatial position we assume the probability for observing a pulse to be $\alpha$ with $0 \leq \alpha \leq 1$. For a long time record of duration $\mathscr{T}$, we have $\alpha = \mu\Delta\tau$ where $\Delta\tau$ is the duration of the pulse and $\mu \approx \langle N \rangle / \mathscr{T}$ is the density of the flux-pulses in the record. Uncertainties due to end-effects can be made arbitrarily small by increasing $\mathscr{T}$. Our analysis addresses the plasma flux signal, but the arguments are readily applicable also for a density signal as discussed by other authors [2].

The probability density for the plasma flux in this basic model is

$$P(\Gamma) = (1-\alpha)\delta(\Gamma) + \alpha\delta(\Gamma - \gamma)\,, \tag{1}$$

where the first term accounts for the cases where the flux vanishes, i.e. at times where no plasma burst is intercepted at the selected position. For the case given by Eq. (1) we readily obtain $\langle \Gamma^m \rangle = \alpha\gamma^m$ with $m = 1, 2, 3, \ldots$, giving the average value $\langle \Gamma \rangle = \alpha\gamma$, the variance $\sigma^2 \equiv \langle (\Gamma - \langle \Gamma \rangle)^2 \rangle = \alpha(1-\alpha)\gamma^2$, the skewness $S = \alpha(1 - 3\alpha + 2\alpha^2)\gamma^3/\sigma^3$ and kurtosis $K = \alpha(1 - 4\alpha + 6\alpha^2 - 3\alpha^3)\gamma^4/\sigma^4$, implying here the exact relation $K = S^2 + 1$ for all $\alpha$. For this particular signal we have $S = 0$ when $\alpha = 1/2$. It can be demonstrated analytically that $K \geq S^2 + 1$ for any signal with $\sigma \neq 0$ so the present simple model represents a limiting case. For the simple two-level model it is not essential that all structures have the same duration, and we can have $\Delta\tau$ to be statistically distributed.

The model Eq. (1) can be generalized by allowing for a general temporal form of the burst event containing a random amplitude parameter $A$ with some probability density $P_A(A)$. In principle the amplitude can assume both signs even though the average is different from zero. Taking $A$ as an amplitude factor we omit cases where the shape of the structure can depend on its magnitude, but the analysis can be extended to include also such cases. We denote a general pulse form by $G_A(\tau)$ for $0 \leq \tau \leq \Delta\tau$ and $G_A(\tau) = 0$ otherwise. We take $\Delta\tau$ to be the same for all pulses, irrespective of $A$. One example could be $G_A(\tau) = A\sin^p(\pi\tau/\Delta\tau)$, where $p$ is a deterministic fitting parameter. In this generalization we find the flux probability density to be

$$P(\Gamma) = (1-\alpha)\delta(\Gamma) + \frac{\alpha}{\Delta\tau}\int_{-\infty}^{\infty}\int_{0}^{\Delta\tau}\delta(\Gamma - G_A(\tau))d\tau P_A(A)dA\,. \tag{2}$$

For the “top-hat” pulses assumed in Eq. (1) the amplitude probability density will be the same as $P_A(A)$, but in general the two PDF’s are different.

Results for $\langle\Gamma\rangle$, $\sigma^2$, $S$ and $K$ are readily obtained by using Eq. (2). Analytically we find, for instance, $\langle\Gamma^m\rangle = (\alpha/\Delta\tau)\int_{-\infty}^{\infty}\int_0^{\Delta\tau} G_A^m(\tau)d\tau P_A(A)dA$. The values of skewness and kurtosis follow approximate parabolic relations for varying parameters for a wide range of pulse shapes and basic statistical distributions, see Fig. 3. We have taken all pulse amplitudes $A$ to be equal. If we allow for a statistical distribution of $A$, we find that the curves retain a parabolic form, but for a given $S$-value, the kurtosis is found to increase, in general.

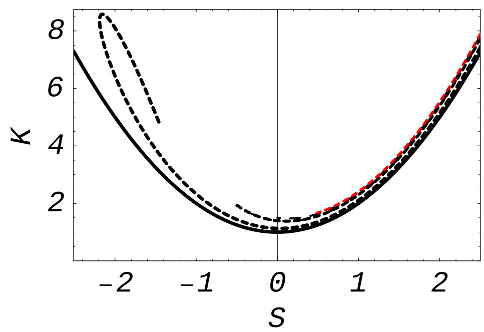


Figure 3: *A parametric presentation of skewness versus kurtosis $(S(\alpha), K(\alpha))$ with varying $\alpha$ for selected model pulses. The heavy solid line gives the “top hat” model for reference, while the first heavy dotted line has $G(\tau) = \sin^p(\pi\tau/\Delta\tau)$ with $p = 1/4$, the next $p = 1/2$, the thinner dashed line has $p = 1$, the next thin dotted line $p = 2$. A red line for $p = 4$ will be almost indistinguishable from this, but it starts for a slightly larger S-value.*

The simplest statistical analysis concerns the lowest order average quantities. These can be determined experimentally without requiring estimates for the full probability density. For long data sequences as available in Blaamann, we can also obtain acceptable estimates for the full probability density of fluctuating quantities, the plasma flux in particular. The model Eq. (2) allows the probability density to be obtained analytically, where we show illustrative results in Fig. 4 for two parameter values. The result agrees well with observations [1]. The model can be generalized to include an additive random noise component, and a transition to a Gaussian random process can be demonstrated in the limit of large noise amplitudes, where in particular the skewness-kurtosis relation degenerates to a point $(K,S) = (3,0)$. The primary effect of a small noise level is to smooth-out the cusp at the origin, see Fig. 4.

Our results differ from a previous study based on correlated Gaussian density and velocity signals without structures present [5]. Although the models are fundamentally different, also the correlated Gaussian model gives a nearly parabolic relation between skewness and kurtosis as in Fig. 2. The flux probability density predicted by the latter model appears similar to ours shown in Fig. 4, notably by having a "cusp" at the origin and two asymmetric "wings" for large positive and negative flux-values. The Gaussian model implicitly assumes that the anomalous transport is due to a random walk of charged particles across magnetic field lines. Our structure-based model assumes the transport to be caused by random bursts of varying amplitude.

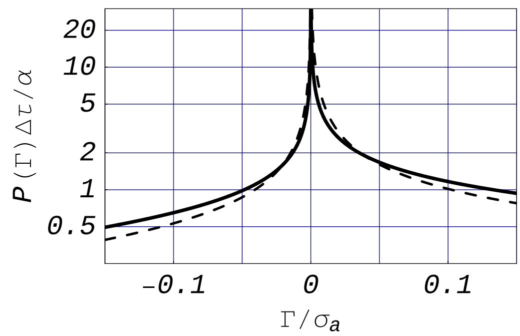


Figure 4: *Flux probability densities for signals constructed by a distribution of pulses* $a\sin^p(\pi\tau/\Delta\tau)$ *with* $p=2$ *(full line) and* $p=4$ *(dashed line). The flux amplitude probability density is* $\exp(-(a-\langle a\rangle)^2/\sigma_a^2)/\sqrt{\pi\sigma_a^2}$. *For the cases shown here we used* $\langle a\rangle/\sigma_a = 0.35$.

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