Dispersion relations for resistive wall modes in tokamaks

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Introduction. There is a long-standing opinion that the ideal MHD cannot describe the RWM (resistive wall mode) dynamics in the DIII-D tokamak [1, 2]. This applies even to the robust rotational stabilization [1] that makes plasma stable essentially above the no-wall stability limit predicted by the ideal MHD. Till now no convincing explanation allowing reliable extrapolation to ITER has been proposed, though some computations show [3, 4] that complete RWM stabilization may be possible due to plasma rotation alone within the standard MHD frame.

With various theoretical attempts, thorough experimental studies and impressive results from DIII-D, the uncertainty in the underlying physics illustrated by detailed analysis [2] of a wide spectrum of competing models still remains unresolved, see also [5].

Facing a necessity of expanding the search beyond the limits of conventional MHD, it is natural to build a wider theory by introducing new elements as additions within the established MHD approach, both physically and mathematically. The latter means, in particular, to use the energy principle algorithm [6] as a guide in derivations. Such a strategy has been outlined in [5, 7]. We follow it here with a focus on the dispersion relations for RWMs in tokamaks.

If the outer region with a resistive wall is described as proposed in [8], the resulting dispersion relation for slow modes will be in the form first appeared in [8],

$$\gamma\tau_D = -W_{no}/W_{iw}, \tag{1}$$

but with additional terms due to non-MHD mechanisms [5]. The representation of the magnetic perturbation $\mathbf{b}$ in vacuum in the Haney-Freidberg (HF) method [8] is the approximation that has never (except [9]) been analyzed for consistency. Here, the limitations of the plasma-wall HF electromagnetic coupling [8] are exposed and better solutions proposed. In (1), γ is the real growth rate of $\mathbf{b}$ assumed to vary in time as $\exp(\gamma t)$, W_{iw} and W_{no} are the perturbed energies with and without an ideal wall, and τ_D is given by Eq. (66) in [8]. In the cylindrical approximation [5, 9],

$$2m\tau_D \equiv (1 - x_{pl}^{2m})\tau_w, \tag{2}$$

where $\tau_w \equiv \mu_0 \sigma d_w b_w$ (b_w, d_w and σ are the wall minor radius, thickness and conductivity), $x_{pl} \equiv b_{pl}/b_w$ with b_{pl} the plasma minor radius, and m the poloidal mode number.

One generalization of (1) is known, it is so-called kinetic dispersion relation [10]

$$\gamma\tau_D = -(W_{no} + \delta W_k)/(W_{iw} + \delta W_k)\,. \tag{3}$$

Below we derive its replacement covering a much wider area and free from its shortcomings.

It is needed because of a number of reasons discussed in [2] and [5]. One of them is that the assumptions used on the way to (3) greatly restrict its applicability range, which fact is ignored in the approach presented in [10]. However, strong impact of hidden constraints on (3) is evident even without calculations. Larger plasma-wall distance makes $W_{no} \to W_{iw}$. Then (3) with (2) predicts $\gamma\tau_D = -1$ or unconditional stability for any plasma. This strikingly unphysical result proves that (3) completely fails at small x_{pl}, and so it does at $x_{pl} \to 1$ too. A search for a narrow window in x_{pl} for formal justification of (3) has never been done in [10] and related studies before. Therefore, reliability of their results based on (3) is, at least, questionable.

Mathematics. A well-established procedure [6, 7] to derive the energy principle, preceding (1) and (3), starts from the linearized force balance equation

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}_{id}(\boldsymbol{\xi}) + \mathbf{F}_{non}(\boldsymbol{\xi}) \tag{4}$$

(called sometimes an equation of small oscillations) for the displacement $\boldsymbol{\xi}$ from equilibrium. Here ρ_0 is the unperturbed plasma mass density, $\mathbf{F}_{non}$ describes the non-ideal part of the force and/or plasma rotation,

$$\mathbf{F}_{id} = -\nabla\tilde{p} + \mathbf{j}_0 \times \mathbf{b} + \tilde{\mathbf{j}} \times \mathbf{B}_0 \tag{5}$$

is the usual MHD force operator with $\mathbf{j}_0$ and $\mathbf{B}_0$ the equilibrium current density and magnetic field, $\mathbf{b}$ and $\tilde{\mathbf{j}} = \nabla \times \mathbf{b}/\mu_0$ the small perturbations, $\tilde{p}$ the pressure perturbation. The operation

$$K_{\eta\xi} \equiv \int_{pl} \frac{\rho_0}{2} \boldsymbol{\eta} \cdot \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} dV = \frac{1}{2}\int_{pl} \boldsymbol{\eta} \cdot (\mathbf{F}_{id} + \mathbf{F}_{non}) dV \equiv -\left(W_{MHD}(\boldsymbol{\xi},\boldsymbol{\eta}) - \oint_{pl} \frac{\mathbf{Q} \times \mathbf{b}_e}{2\mu_0} \cdot d\mathbf{S} \right) - W_{non}(\boldsymbol{\xi},\boldsymbol{\eta}) \tag{6}$$

with a properly selected function $\boldsymbol{\eta}$ ($= \boldsymbol{\xi}$, $\dot{\boldsymbol{\xi}}$, $\boldsymbol{\xi}^*$ or $\dot{\boldsymbol{\xi}}^*$, the dot denotes the time derivative) and $\mathbf{Q}(\boldsymbol{\eta})$ (respectively, the vector potential $\mathbf{A}$, $\dot{\mathbf{A}}$, $\mathbf{A}^*$ or $\dot{\mathbf{A}}^*$) gives a variety of integral energy relations used in theory. In [7], full section VI is devoted to transformations of (6).

With substitutions $\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)$, $\tilde{p} = \tilde{p}_{id} \equiv -\boldsymbol{\xi} \cdot \nabla p_0 - \Gamma p_0 \,\mathrm{div}\,\boldsymbol{\xi}$ and $\mathbf{B}_0 \cdot d\mathbf{S}_{pl} = 0$ (Γ is the ratio of the specific heats), $\boldsymbol{\eta} = \boldsymbol{\xi}^*$ and, accordingly, $\mathbf{Q} = \mathbf{A}^*$, (6) reduces to

$$K + W_p + W_s + W_{non} = \frac{1}{2\mu_0} \oint_{pl} \varphi_e \mathbf{b}_e^* \cdot d\mathbf{S} = \frac{1}{2\mu_0} \oint_{pl} (\mathbf{A}_e^* \times \mathbf{b}_e) \cdot d\mathbf{S}\,, \tag{7}$$

where we have used $\mathbf{A}^* \times \nabla\varphi = -\nabla \times (\varphi \mathbf{A}^*) + \varphi \mathbf{b}^*$. The subscript e means external (to the plasma), for other definitions see (33)–(40) in [7]. The whole effect of the external world is accumulated in the surface integrals.

In the HF approach [8] revised and extended in [5, 9], $\mathbf{b} \cdot \mathbf{n}_{pl}$ ($\mathbf{n}_{pl}$ is a unit normal to the plasma) is considered a given function, and the magnetic field $\mathbf{b}$ in the plasma-wall vacuum gap and behind the wall is approximated, as in a circular cylinder, by

$$\mathbf{b}_{HF} = \nabla\varphi_{HF} = \nabla \times \mathbf{A}_{HF} = c_{no}\mathbf{b}_{no} + c_{iw}\mathbf{b}_{iw} \tag{8}$$

with $\mathbf{b}_{no} = \nabla\varphi_{no}$ and $\mathbf{b}_{iw} = \nabla\varphi_{iw}$ the no-wall and ideal-wall solutions of $\nabla^2\varphi = 0$ such that $\mathbf{n}_{pl} \cdot \nabla\varphi = \mathbf{n}_{pl} \cdot \mathbf{b}$ for each function (and $\mathbf{n}_w \cdot \nabla\varphi_{iw} = 0$). Then $\mathbf{n}_{pl} \cdot (\mathbf{b} - \mathbf{b}_{HF}) = 0$ is satisfied by

$$c_{no} + c_{iw} = 1. \tag{9}$$

The validity of (8) has never been analysed. Therefore, we have to keep $\varphi_e \neq \varphi_{HF}$. Then

$$-\oint_{pl} \varphi_e \mathbf{b}_e^* \cdot d\mathbf{S} = 2\mu_0[(c_{no}N + c_{iw}I) + \alpha_\varphi], \tag{10}$$

where we have used $\mathbf{b}_{no} = \nabla\varphi_{no}$ defined in the whole space *out* outside the plasma, while

$$2\mu_0 N \equiv \int_{out} |\mathbf{b}_{no}|^2 dV, \qquad 2\mu_0 I \equiv \int_{gap} |\mathbf{b}_{id}|^2 dV, \qquad 2\mu_0\alpha_\varphi \equiv \oint_{pl} (\varphi_{HF} - \varphi_e)\mathbf{b}_e^* \cdot d\mathbf{S}. \tag{11}$$

Substitution of (10) into (7) yields, with account of (9),

$$c_{iw}(W_{iw} + W_{add}) + c_{no}(W_{no} + W_{add}) = 0, \tag{12}$$

where $W_{no} \equiv W_p + N$ and $W_{iw} \equiv W_p + I$ are the same conventional MHD integrals as in (1), and

$$W_{add} \equiv K + W_s + W_{non} + \alpha_\varphi. \tag{13}$$

To get a dispersion relation, we have to incorporate the dependence of the right-hand side in (7) on the current induced in the wall. For this purpose, one can use the equality

$$\oint_{pl} (\mathbf{A}_e^* \times \mathbf{b}_e) \cdot d\mathbf{S} = -\int_{out} |\mathbf{b}_e|^2 dV + \mu_0 \int_{wall} \mathbf{A}_e^* \cdot \mathbf{j} dV, \tag{14}$$

supplemented by the Ohm's law for the wall, $\mathbf{j} = \sigma\mathbf{E}_\parallel$ with $\mathbf{E} = -\dot{\mathbf{A}}$ reducing to $\mathbf{E} = -\gamma\mathbf{A}$ for perturbations varying in time as $\exp(\gamma t)$, as in the HF approach [8]. According to (7),

$$-\oint_{pl} (\mathbf{A}_{HF}^* \times \mathbf{b}_{HF}) \cdot d\mathbf{S} = 2\mu_0[c_{no}N + c_{iw}I - c_{no}^* c_{iw}(I - N)] + \gamma\mu_0 \int_{wall} \sigma |\mathbf{n}_w \times \mathbf{A}_{HF}|^2 dV \tag{15}$$

must be equal to the right-hand side of (10) at $\alpha_\varphi = 0$. This gives us

$$c_{iw} = c_{no}\gamma\tau_D \tag{16}$$

with time constant (greatly different from the resistive wall time $\tau_w \equiv \mu_0 \sigma d_w b_w$)

$$\tau_D \equiv \frac{1}{2(I-N)} \int_{wall} \sigma |\mathbf{n}_w \times \mathbf{A}_{no}|^2 dV \,. \tag{17}$$

By using (16) we obtain from (12) a replacement of (1) and (3):

$$\gamma\tau_D = -(W_{no} + W_{add})/(W_{iw} + W_{add}) \,. \tag{18}$$

While W_{no} and W_{iw} are familiar real functions of (in general – complex) $\boldsymbol{\xi}$, W_{add} is a new element depending on $\mathbf{F}_{non}$ in (4), on $\mathbf{b} - \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)$, the non-MHD effects that make $\tilde{p} \neq \tilde{p}_{id}$, and on the difference $\varphi_e - \varphi_{HF}$ resulting in $\alpha_\varphi \neq 0$, see (11) and one example in [9].

Discussion. Note that φ_{HF} is a cylindrical ansatz, its applicability to real toroidal tokamaks is debatable even without mentioning noncircular plasma and wall. The largest error for the use of (1) and (3) obtained with $\mathbf{b}_{HF}$ replacing $\mathbf{b}$ should be expected for compact devices like NSTX.

According to (18), at real W_{add} (simplest case) the stability boundary must be given by

$$W_{no} + W_{add} = 0 \tag{19}$$

instead of $W_{no} = 0$ predicted by (1). The fact that the plasma can be stable in the region where $W_{no} < 0$ implies that W_{add} can be large and positive. Separate studies are needed to find what can make $W_{add} > 0$ and to what extent. Equally or even more important must be $\mathrm{Im} W_{add}$.

Relation (18) derived from the first principles explains why, in the absence of studies allowing to identify or, at least, distinguish the dominating contributions into W_{add} or $\mathbf{F}_{non}$, the statement "RWM stabilization is best accomplished by a combination of passive *kinetic* stabilization …" in [11] is wrong. It is based on tenuous use of the HF approach for compact device NSTX, and the claimed "success" [10] is a product of undisclosed manipulations with (3) involving ad hoc selections of parameters. The "kinetic" mechanism is not convincingly demonstrated, the alternative variants have neither been analyzed nor ruled out.

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