AN IMPROVED FORMULATION OF RIPPLE-AVERAGED KINETIC THEORY

C.D. Beidler, H. Maaßberg
Max-Planck-Institut für Plasmaphysik
IPP–EURATOM Association
85748 Garching bei München, GERMANY

Neoclassical transport for stellarators in the long-mean-free-path *(lmfp)* regime is often estimated using the results of *ripple-averaged* kinetic theory (more commonly referred to as *bounce-averaged* kinetic theory when only particles localized in the stellarator's helical ripple are considered). In its usual form, this theory provides a reasonably accurate description of classical helical devices such as LHD and W7-A, however its predictions for the W7-X device and various quasi-isodynamic configurations have been found to be significantly at odds with those of numerical simulations. The reason for this discrepancy can be traced to several simplifying assumptions made during the ripple average, the most critical of which is the assumption of "small" rotational transform per field period. A formulation of the ripple-averaged kinetic theory which removes the shortcomings of the conventional approach is described below.

The linearized drift kinetic equation $(r, \zeta = N\phi, \theta, \mathcal{E}, \mu)$ may be written

$$\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\partial f_m}{\partial r} + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\frac{\partial f_1}{\partial \zeta} + \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\partial f_1}{\partial \theta} = C(f_1)$$

Here, (r, ϕ, θ) are Boozer's toroidal flux coordinates identified with the flux surface, toroidal angle and poloidal angle, respectively, N is the field period number, $\mathcal{E}$ the total energy and $\mu = mv_\perp^2/2B$ the magnetic moment. The total distribution function, $f_m + f_1$, is described by a lowest-order local Maxwellian plus a first-order perturbation. The collision operator, C, is given by the Lorentz model. The drift equations are

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial \theta} \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = N\,\frac{\lambda v B}{g}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \iota\,\frac{\lambda v B}{g} - \frac{1}{rB_0}\frac{\partial \Phi}{\partial r} - \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial r}$$

with B the magnetic field strength, $\Phi = \Phi(r)$ the electrostatic potential, $v_d = \kappa/qR_0B_0$ the ∇B drift velocity, $\kappa = mv^2/2$ the kinetic energy, q the charge, R_0 the major radius, B_0 the average value of B at $r = 0$, $\epsilon_t = r/R_0$ the inverse aspect ratio, $\lambda = v_{||}/v$ the pitch angle, ι the rotational transform and g the "poloidal current" $(g \approx R_0B_0)$.

The magnetic field strength is given by the *multiple-helicity* model [1]

$$B/B_0 = 1 + b_{0,0} + b_{1,0}\cos\theta + b_{2,0}\cos 2\theta - \epsilon_H\cos(\zeta - \chi)$$

$$\epsilon_H = \left(\mathcal{C}^2 + \mathcal{S}^2\right)^{1/2} \qquad -\cos\chi = \mathcal{C}/\epsilon_H \qquad -\sin\chi = \mathcal{S}/\epsilon_H$$

$$\mathcal{C} = b_{0,1} + \sum_{m=1}^{\infty}(b_{m,1} + b_{m,-1})\cos m\theta \qquad \mathcal{S} = \sum_{m=1}^{\infty}(b_{m,1} - b_{m,-1})\sin m\theta$$

which assumes that B is adequately described by the $n = -1, 0, 1$ terms in the full Fourier $b_{m,n}$ decomposition.

For electrons and (usually) for bulk-plasma ions the drift equations describe motion on two quite different time scales, the rapid motion of particles along field lines (with velocity λv) and the much slower drifts off the same (with velocities v_d and $(\partial\Phi/\partial r)/B_0$). Further, in the *lmfp* regime where collisions are rare, the rapid motion may be considered (nearly) periodic with respect to the local helical ripple, ϵ_H. This situation makes it possible to consider a time average of the drift kinetic equation with the goal of reducing the number of phase space variables in the problem.

Unfortunately, the drift equations in the coordinate system (r, ζ, θ) do not allow a straightforward separation of the time scales. In particular, $\mathrm{d}\theta/\mathrm{d}t$ contains both "fast" and "slow" drift terms. This problem is commonly "solved" by taking ι/N to be small (in practice, $\iota/N = 0$ is assumed), i.e. r and θ and all functions of these two variables are considered constants with respect to the time average. Further consequences are that the average is carried out along ζ (and not along the field line) and that all local ripples are taken to be symmetric so that only localized and "locally passing" particle orbits exist.

A more elegant means for dealing with the separation of time scales is to introduce a change of variables. For the local ripple with index i one defines $\theta = \theta_n^i + (\iota/N)(\zeta - \zeta_n^i)$, where $\theta = \theta_n^i$, $\zeta = \zeta_n^i$ specifies the local minimum of B. The new variable, θ_n has two important properties: (1) for the multiple-helicity model B every local ripple is uniquely defined by its value of θ_n; and (2) θ_n and functions of θ_n are truly constant within the domain of a single ripple (and therefore truly constant with respect to the time average) except for the discontinuous "jump" at each local maximum of B, i.e. in passing from ripple i to ripple $i \pm 1$. The drift equations in the new coordinate system (r, ζ, θ_n) are

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left(1 - \frac{\iota}{N}\frac{\partial\zeta_n}{\partial\theta_n}\right)^{-1} \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial\theta_n} \qquad\qquad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = N\,\frac{\lambda v B}{g}$$

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = (\Delta\theta_n)^i\,\delta_D\left(\zeta - \zeta_x^i\right) N\,\frac{\lambda v B}{g} - \left(1 - \frac{\iota}{N}\frac{\partial\zeta_n}{\partial\theta_n}\right)^{-1}\left\{\frac{1}{rB_0}\frac{\partial\Phi}{\partial r} + \frac{v_d}{\epsilon_t}(1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial r}\right\}$$

where δ_D is the Dirac delta function and $\Delta\theta_n$ the change in the value of θ_n associated with passing the local maximum in B at $\zeta = \zeta_x^i$.

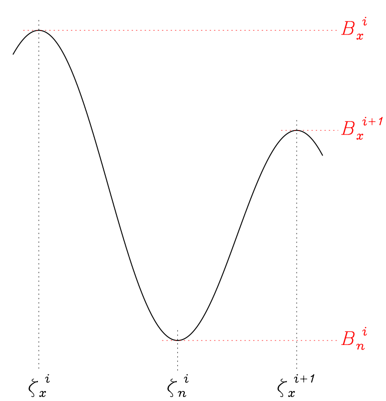
Next, the magnetic field strength must be expressed in the new coordinates. This is most conveniently done by introducing a local model field which consists of two "half" sinusoids (with indices $j = 1, 2$) with extrema identical to those of the actual field. Explicitly

$$B_j^i = B_n^i + B_0\delta_j^i(1 - \cos\eta_j^i)$$

$$\delta_1^i = \frac{B_x^i - B_n^i}{2B_0} \qquad \delta_2^i = \frac{B_x^{i+1} - B_n^i}{2B_0}$$

$$\eta_j^i = \frac{\zeta - \zeta_n^i}{L_j^i}$$

$$L_1^i = \frac{(\zeta_n^i - \zeta_x^i)}{\pi} \qquad L_2^i = \frac{(\zeta_x^{i+1} - \zeta_n^i)}{\pi}$$



with $-\pi \le \eta_1^i \le 0$ and $0 \le \eta_2^i \le \pi$. All extrema are determined numerically as analytic approximations have been found to introduce unacceptable error.

The drift-kinetic equation may now be time averaged using the operator

$$\langle x_j \rangle = \frac{1}{\tau}\frac{g}{NB_0}\left\{ L_1 \int_{-\eta_1^*}^{0} \frac{\mathrm{d}\eta_1}{u_1}\frac{B_0}{B_1} x_1 + L_2 \int_{0}^{\eta_2^*} \frac{\mathrm{d}\eta_2}{u_2}\frac{B_0}{B_2} x_2 \right\} \qquad \eta_j^* = \begin{cases} 2\sin^{-1} k_j & k_j^2 \le 1 \\ \pi & k_j^2 \ge 1 \end{cases}$$

which has been constructed to annihilate the $(\mathrm{d}\zeta/\mathrm{d}t)(\partial f_1/\partial\zeta)$ term in the kinetic equation and eliminate ζ as a variable from the problem. In the operator, the time τ is defined so that $\langle 1 \rangle = 1$ and

$$u_j \equiv |\lambda_j v| = v\left(2\delta_j \frac{\mu B_0}{\kappa}\right)^{1/2}\left(k_j^2 - \sin^2\frac{\eta_j}{2}\right)^{1/2} \qquad k_j^2 = \frac{\kappa/\mu - B_n}{2B_0\delta_j}$$

The resulting *ripple-averaged* kinetic equation may be written

$$\left\langle \frac{\mathrm{d}r}{\mathrm{d}t} \right\rangle \frac{\partial f_m}{\partial r} + \left\langle \frac{\mathrm{d}\theta_n}{\mathrm{d}t} \right\rangle \frac{\partial f_1}{\partial \theta_n} + \left\langle \frac{\mathrm{d}k^2}{\mathrm{d}t} \right\rangle \frac{\partial f_1}{\partial k^2} = \left\langle C(f_1) \right\rangle$$

where the drift equations are

$$\left\langle \frac{\mathrm{d}r}{\mathrm{d}t} \right\rangle = \left(1 - \frac{\iota}{N}\frac{\partial\zeta_n}{\partial\theta_n}\right)^{-1} \frac{v_d}{\epsilon_t}\left\langle (1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial\theta_n} \right\rangle$$

$$\left\langle \frac{\mathrm{d}\theta_n}{\mathrm{d}t} \right\rangle = \frac{\Delta\theta_n}{\tau} H(k^2-1) - \left(1 - \frac{\iota}{N}\frac{\partial\zeta_n}{\partial\theta_n}\right)^{-1} \left\{ \frac{1}{rB_0}\frac{\partial\Phi}{\partial r} + \frac{v_d}{\epsilon_t}\left\langle (1+\lambda^2)\frac{1}{B}\frac{\partial B}{\partial r} \right\rangle \right\}$$

$$\left\langle \frac{\mathrm{d}k^2}{\mathrm{d}t} \right\rangle = \frac{\partial k^2}{\partial r}\left\langle \frac{\mathrm{d}r}{\mathrm{d}t} \right\rangle + \frac{\partial k^2}{\partial\theta_n}\left\langle \frac{\mathrm{d}\theta_n}{\mathrm{d}t} \right\rangle$$

and where H is the Heaviside function. The pitch-angle variable $k^2 = (\kappa/\mu - B_n)/2B_0\delta$ with $\delta = \min(\delta_1, \delta_2)$ has been defined so that all localized particles satisfy $0 \le k^2 \le 1$. After averaging, the ripple index i becomes superfluous and may therefore be dropped.

The Asymptotic $1/\nu$ Regime

To illustrate the efficacy of the improved theory, consider the asymptotic $1/\nu$ regime. The kinetic equation to be solved here (in normalized form) is

$$\frac{1}{v_d}\left\langle \frac{\mathrm{d}r}{\mathrm{d}t} \right\rangle = \frac{2\epsilon_h}{\nu}\left\langle C(\widehat{f}_1) \right\rangle \qquad \widehat{f}_1 = f_1\left(2\epsilon_h \frac{v_d}{\nu}\frac{\partial f_m}{\partial r}\right)^{-1}$$

with ν the 90-degree deflection frequency and ϵ_h the average value of ϵ_H (the explicit definition is given below). This equation is solved numerically with the boundary condition that $\widehat{f}_1 = 0$ for non-localized particles. The solution may then be used to determine the mono-energetic diffusion coefficient, D. For results presented here, a normalized form is chosen $\widehat{D} = D/D_{SH}$, where D_{SH} is the diffusion coefficient expected from the conventional theory [1]. Explicitly

$$D_{SH} = \frac{4}{9\pi^2}\left(\frac{v_d}{\epsilon_t}\right)^2 \frac{I_{SH}}{\nu} \qquad I_{SH} = \int_0^{2\pi} \mathrm{d}\theta\, (2\epsilon_H)^{3/2} \left\{ \left(\frac{\partial\epsilon_T}{\partial\theta}\right)^2 - \frac{6}{5}\frac{\partial\epsilon_T}{\partial\theta}\frac{\partial\epsilon_H}{\partial\theta} + 0.385\left(\frac{\partial\epsilon_H}{\partial\theta}\right)^2 \right\}$$

with $\epsilon_T = b_{1,0}\cos\theta + b_{2,0}\cos 2\theta$.

As an example, the W7-X device is considered. For W7-X it is possible to vary the toroidal mirror term in the Fourier representation of B in the range $0 \leq b_{0,1}(r=0) \leq 0.1$ while leaving the other harmonics essentially unchanged (the standard configuration has $b_{0,1} \approx 0.046$ on the magnetic axis). The effect of this variation on $1/\nu$ transport and the importance of avoiding the assumption of small ι/N are illustrated in Figure 1 by plots of the effective helical ripple (ϵ_{eff}; also shown for comparison is the average helical ripple, ϵ_h) and $\widehat{D}$, respectively, where

$$\epsilon_h \equiv \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta\, \epsilon_H \qquad\qquad \epsilon_{eff} = \frac{1}{2} \left(\frac{\widehat{D}\, I_{SH}}{\pi \epsilon_t^2} \right)^{2/3}$$

Of the three W7-X configurations investigated, only the one with $b_{0,1}(0) = 0$ is adequately described by the conventional theory. For the other two, $\widehat{D}$ varies considerably in the range $1/3 < \widehat{D} < 3$. Also shown are results for a "linked mirror" with $b_{0,1}(0) \approx 0.2$ [2]. In spite of this very large ripple, the configuration has extremely small values of ϵ_{eff} near the magnetic axis. The conventional theory is particularly poor in this case, overestimating the diffusion coefficient by an order of magnitude.

All results obtained with the improved formulation of ripple-averaged kinetic theory have been verified using the Drift Kinetic Equation Solver (DKES) [3]. Solutions of the ripple-averaged kinetic equation typically require only 0.1% of the computational resources consumed by DKES.

[1] Shaing K C and Hokin S A, 1983 **Phys. Fluids 26**, 2136.

[2] Dommaschk W, Herrnegger F and Schlüter A, 1994 **Proc. 21st EPS Conf. Control. Fusion and Plas. Phys.** , Part I, 360.

[3] van Rij W I and Hirshman S P, 1989 **Phys. Fluids B 1**, 563.

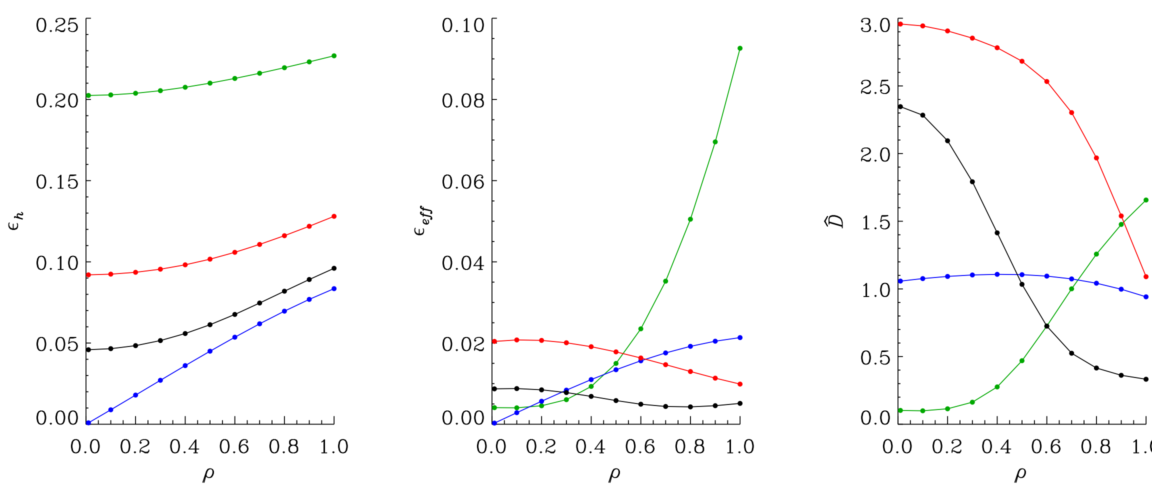


Figure1. The average helical ripple, effective helical ripple and normalized diffusion coefficient are plotted as functions of normalized radius for a "linked mirror" device and for three W7-X configurations with $b_{0,1}(0) = 0$, $b_{0,1}(0) \approx 0.046$ and $b_{0,1}(0) = 0.092$.